

THE DEFECT OF WEAK APPROXIMATION FOR HOMOGENEOUS SPACES. II

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ABSTRACT. Let X be a right homogeneous space of a connected linear algebraic group G' over a number field k , containing a k -point x . Assume that the stabilizer of x in G' is connected. Using the notion of a quasi-trivial group, recently introduced by Colliot-Thélène, we can represent X in the form $X = H \backslash G$, where G is a quasi-trivial k -group and $H \subset G$ is a connected k -subgroup.

Let S be a finite set of places of k . Applying results of [B2], we compute the defect of weak approximation for X with respect to S in terms of the biggest toric quotient H^{tor} of H . In particular, we show that if H^{tor} splits over a metacyclic extension of k , then X has the weak approximation property. We show also that any homogeneous space X with connected stabilizer (without assumptions on H^{tor}) has the real approximation property.

1. INTRODUCTION

This note is a sequel for [B2], and we use the notation of that paper. Let k be a number field, and let \bar{k} be a fixed algebraic closure of k . We write \mathcal{V} for the set of all places of k , and \mathcal{V}_∞ for the set of its archimedean places. If $v \in \mathcal{V}$, we write k_v for the completion of k at v .

Let X be an algebraic variety over k . We refer to [B2] for preliminaries on weak approximation for X . If $S \subset \mathcal{V}$ is a finite set of places, we write (WA_S) for the weak approximation property with respect to S . Thus, “ X has (WA_S) ” means that $X(k)$ is dense in $\prod_{v \in S} X(k_v)$. We say that X has *the weak approximation property*, if X has (WA_S) for any finite subset $S \subset \mathcal{V}$. We say that X has *the real approximation property*, if X has the weak approximation property (WA_S) with respect to $S = \mathcal{V}_\infty$.

In [B2] we considered the case $X = H \backslash G$, where $H \subset G$ is a connected k -subgroup of a connected k -group G , assuming that $\text{III}(G) = 0$ and $A(G) = 0$ (the assumption $A(G) = 0$ means that G has the weak approximation property). Under these assumptions we constructed a certain abelian group $C_S(H, G)$ which is the defect of weak approximation for X with respect to S : the variety X has (WA_S) if and only if $C_S(H, G) = 0$. We initially

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constructed $C_S(H, G)$ in terms of H and G , but then we computed it in terms of the Brauer group of X .

In the present note we consider the case of an arbitrary homogeneous space with connected stabilizer $X = H' \backslash G'$, where G' is any connected linear k -group and $H' \subset G'$ is a connected k -subgroup. Using the notions of a quasi-trivial k -group and a flasque resolution, introduced by J.-L. Colliot-Thélène [CT], we notice that we can represent X in the form $X = H \backslash G$, where G is a *quasi-trivial* group and $H \subset G$ is a connected k -subgroup (Lemma 2.5). We have $\text{III}(G) = 0$ and $A(G) = 0$, because G is quasi-trivial. Now we can apply [B2, Theorem 1.3]. We obtain that X has (WA_S) if and only if $C_S(H, G) = 0$.

Moreover, we have $\text{Pic}(G_K) = 0$ for any field extension K/k , because G is quasi-trivial. Using this fact, we show that the group $C_S(H, G)$ can be computed in terms of H only. Namely, we construct a group $C_S(H)$ in terms of H as in [B1] and prove that $C_S(H, G) = C_S(H)$ (Lemma 3.4).

We see that X has (WA_S) if and only if $C_S(H) = 0$. We say that $C_S(H)$ is the *defect of weak approximation for X with respect to S* . Note that the group $C_S(H)$ does not depend on the representation of X in the form $X = H \backslash G$ with quasi-trivial G and connected H , because it can be computed in terms of the Brauer group of X ([B2, Theorem 1.11]).

Let H^{tor} denote the biggest quotient torus of H . We show that the canonical homomorphism $C_S(H) \rightarrow C_S(H^{\text{tor}})$ is an isomorphism (Proposition 3.7). It follows that X has (WA_S) if and only if $C_S(H^{\text{tor}}) = 0$. We notice that

$$C_S(H^{\text{tor}}) \simeq \text{coker} \left[H^1(k, H^{\text{tor}}) \rightarrow \prod_{v \in S} H^1(k_v, H^{\text{tor}}) \right].$$

Let L/k be a Galois extension splitting the torus H^{tor} . Let S_0 denote the set of (nonarchimedean, ramified in L) places v of k such that the decomposition group of v in $\text{Gal}(L/k)$ is noncyclic. We prove that $C_S(H) = C_{S \cap S_0}(H)$ (Corollary 3.11).

Assume that $S \cap S_0 = \emptyset$, i.e. all the places in S have cyclic decomposition subgroups in $\text{Gal}(L/k)$. Then $C_S(H) = 0$, hence X has (WA_S) (Theorem 3.12). In particular, $C_{\mathcal{V}_\infty}(H) = 0$ for any H . Thus any homogeneous space X of a connected k -group with connected stabilizer has the real approximation property (Corollary 3.13).

Now assume that H^{tor} splits over a cyclic extension of k (e.g. $H^{\text{tor}} = 1$). Then $S_0 = \emptyset$, hence $C_S(H) = 0$ for any S , and X has the weak approximation property (Corollary 3.14). Moreover, we prove that if H^{tor} splits over a metacyclic extension, then X has the weak approximation property (Theorem 4.2).

These results generalize the results of [B1], where we assumed that G is semisimple simply connected. They also generalize results of Sansuc [Sa] on weak approximation for connected linear groups.

We could state and prove our results thanks to the notion of a quasi-trivial group introduced by Colliot-Thélène [CT]. The constructions and proofs are based on results of Kottwitz [K]. Of course, our results are based on the classical results of Kneser, Harder, Chernousov, and Platonov on the Hasse principle and weak approximation for simply connected semisimple groups.

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2. PRELIMINARIES ON QUASI-TRIVIAL GROUPS

The results of this section are actually due to J.-L. Colliot-Thélène [CT].

2.1. Let k be a field of characteristic 0, \bar{k} a fixed algebraic closure of k . Let G be a connected linear k -group. We set $\bar{G} = G \times_k \bar{k}$. We use the following notation:

G^u is the unipotent radical of G ;

$G^{\text{red}} = G/G^u$ (it is reductive);

G^{ss} is the derived group of G^{red} (it is semisimple);

$G^{\text{tor}} = G^{\text{red}}/G^{\text{ss}}$ (it is a torus);

$G^{\text{ssu}} = \ker[G \rightarrow G^{\text{tor}}]$ (it is an extension of G^{ss} by G^u).

Definition 2.2 (J.-L. Colliot-Thélène). A connected linear k -group G over a field k of characteristic 0 is called *quasi-trivial*, if G^{tor} is a quasi-trivial torus and G^{ss} is simply connected.

Recall that a k -torus T is called quasi-trivial if its character group $\mathbf{X}(\bar{T})$ is a permutation $\text{Gal}(\bar{k}/k)$ -module.

Note that if G is quasi-trivial, then for any field extension K/k the group G_K is quasi-trivial.

Lemma 2.3. *Let G be a quasi-trivial group over a field k of characteristic 0. Then $\text{Pic}(G) = 0$, where Pic denotes the Picard group.*

Proof. If

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

is a short exact sequence of connected linear k -groups, then we have an exact sequence

$$(1) \quad \mathbf{X}(G') \rightarrow \text{Pic}(G'') \rightarrow \text{Pic}(G) \rightarrow \text{Pic}(G'),$$

where $\mathbf{X}(G')$ denotes the group of k -characters of G' , see [Sa, Corollary 6.11].

Since G^u is a unipotent k -group, the exponential map $\exp: \text{Lie } G^u \rightarrow G^u$ is a biregular isomorphism of algebraic varieties (because $\text{char}(k) = 0$), hence $\text{Pic}(G^u) = 0$. By [Sa, Lemme 6.9] $\text{Pic}(G^{\text{ss}}) = 0$ (because G^{ss} is

simply connected) and $\text{Pic}(G^{\text{tor}}) = H^1(k, \mathbf{X}(\overline{G}^{\text{tor}}))$. Since $\mathbf{X}(\overline{G}^{\text{tor}})$ is a permutation module, we see that $\text{Pic}(G^{\text{tor}}) = 0$. Using exact sequence (1), we conclude by dévissage that $\text{Pic}(G) = 0$. \square

Lemma 2.4. *Let G be a quasi-trivial k -group over a number field k . Then $\text{III}(G) = 0$ and $A(G) = 0$.*

Proof. By [CT, Proposition 9.2] we have $\text{III}(G^{\text{red}}) = 0$ and $A(G^{\text{red}}) = 0$. By [Sa, Proposition 4.1] $\text{III}(G) = \text{III}(G^{\text{red}})$. By [Sa, Proposition 3.2] $A(G) = A(G^{\text{red}})$. Thus $\text{III}(G) = 0$ and $A(G) = 0$. \square

Lemma 2.5. *Let k be a field of characteristic 0 and X a right homogeneous space with connected stabilizer over k , i.e. $X = H' \backslash G'$, where G' is a connected linear k -group and $H' \subset G'$ is a connected k -subgroup. Then one can represent X as $X = H \backslash G$, where G is a quasi-trivial k -group and $H \subset G$ is a connected k -subgroup.*

Proof. By [CT, Proposition-Définition 3.1] there exists a flasque resolution of G' , i.e. a central extension of connected k -groups

$$1 \rightarrow F \rightarrow G \rightarrow G' \rightarrow 1,$$

where G is quasi-trivial and F is a flasque k -torus. Let H be the preimage of H' in G . From the exact sequence

$$1 \rightarrow F \rightarrow H \rightarrow H' \rightarrow 1$$

we see that H is connected, because H' and F are connected. We have $X = H \backslash G$. \square

3. DEFECT OF WEAK APPROXIMATION

3.1. Let X be a homogeneous space with connected stabilizer over a number field k , i.e. $X = H' \backslash G'$, where G' is a connected linear k -group and $H' \subset G'$ is a connected k -subgroup. By Lemma 2.5 we may write $X = H \backslash G$, where G is a quasi-trivial k -group and $H \subset G$ is a connected k -subgroup.

By Lemma 2.4 $\text{III}(G) = 0$ and $A(G) = 0$. Therefore we can apply the results of [B2].

3.2. Let X , G , H be as in 3.1. Let $S \subset \mathcal{V}$ be a finite subset. Set

$$\begin{aligned} B(H) &= \text{Hom}(\text{Pic}(H), \mathbb{Q}/\mathbb{Z}) = (\pi_1(H)_{\Gamma})_{\text{tors}} \\ B_v(H) &= B(H_{k_v}) \text{ for } v \in \mathcal{V} \end{aligned}$$

with the notation of [B2]. Consider the canonical homomorphism

$$\lambda_v: B_v(H) \rightarrow B(H).$$

Set:

$$\begin{aligned} B^S(H) &= \langle \lambda_v(B_v(H)) \rangle_{v \in \mathcal{V} \setminus S} \\ B'(H) &= B^{\emptyset}(H) = \langle \lambda_v(B_v(H)) \rangle_{v \in \mathcal{V}} \\ C_S(H) &= B'(H) / B^S(H), \end{aligned}$$

where $\langle \lambda_v(B_v(H)) \rangle_{v \in \mathcal{V} \setminus S}$ denotes the subgroup of $B(H)$ generated by the subgroups $\lambda_v(B_v(H))$ for all $v \in \mathcal{V} \setminus S$.

3.3. For a homogeneous space $X = H \backslash G$ over k , without assuming that G is quasi-trivial, we defined in [B2] the following groups:

$$\begin{aligned} B(H, G) &= \ker[B(H) \rightarrow B(G)], \\ B_v(H, G) &= B(H_{k_v}, G_{k_v}) = \ker[B_v(H) \rightarrow B_v(G)], \end{aligned}$$

and also $B^S(H, G)$, $B'(H, G)$, and $C_S(H, G)$, see [B2, Section 1.2].

Lemma 3.4. *Let k , X , G , H be as in 3.1 (in particular G is quasi-trivial). Then there is a canonical isomorphism $C_S(H, G) \xrightarrow{\sim} C_S(H)$.*

Proof. Since G is quasi-trivial, by Lemma 2.3 $\text{Pic}(G) = 0$, hence $B(G) = 0$. Since G_{k_v} is also quasi-trivial, we see that $B_v(G) = 0$. We obtain successively that $B(H, G) = B(H)$, $B_v(H, G) = B_v(H)$, $B^S(H, G) = B^S(H)$, $B'(H, G) = B'(H)$, whence $C_S(H, G) = C_S(H)$. \square

Theorem 3.5. *Let k , X , G , H be as in 3.1 (in particular G is quasi-trivial). Let $S \subset \mathcal{V}$ be a finite set of places of k . Then X has (WA_S) if and only if $C_S(H) = 0$.*

Proof. By Lemma 2.4 $\text{III}(G) = 0$ and $A(G) = 0$. By [B2, Theorem 1.3] X has (WA_S) if and only if $C_S(H, G) = 0$. By Lemma 3.4 $C_S(H, G) = C_S(H)$, and the theorem follows. \square

Lemma 3.6. *Let H be a connected linear k -group over a number field k . Assume that $H^{\text{tor}} = 1$. Then for any place v of k the map $\lambda_v: B_v(H) \rightarrow B(H)$ is surjective.*

Proof. See [CTX, Proof of Theorem 3.4(b)]. \square

Proposition 3.7 ([B1, Theorem 1.4]). *Let H be a connected k -group over a number field k . Let $S \subset \mathcal{V}$ be a finite set of places of k . Then the canonical homomorphism $C_S(H) \rightarrow C_S(H^{\text{tor}})$ is an isomorphism.*

Proof. Since [B1] is not easily accessible, we reproduce the proof here.

First, consider H^{ssu} . Since $(H^{\text{ssu}})^{\text{tor}} = 1$, by Lemma 3.6 for any place v of k we have $\lambda_v(B_v(H^{\text{ssu}})) = B(H^{\text{ssu}})$. We see that $B^S(H^{\text{ssu}}) = B'(H^{\text{ssu}}) = B(H^{\text{ssu}})$.

Consider the canonical short exact sequence

$$1 \rightarrow H^{\text{ssu}} \rightarrow H \rightarrow H^{\text{tor}} \rightarrow 1.$$

Exact sequence (1) from the proof of Lemma 2.3 gives us an exact sequence

$$\mathbf{X}(H^{\text{ssu}}) \rightarrow \text{Pic}(H^{\text{tor}}) \rightarrow \text{Pic}(H) \rightarrow \text{Pic}(H^{\text{ssu}}),$$

where clearly $\mathbf{X}(H^{\text{ssu}}) = 0$. We obtain the dual exact sequence

$$B(H^{\text{ssu}}) \rightarrow B(H) \rightarrow B(H^{\text{tor}}) \rightarrow 0$$

and similar exact sequences for the groups B_v . Since $B^S(H^{\text{ssu}}) = B(H^{\text{ssu}})$, we obtain an exact sequence

$$B(H^{\text{ssu}}) \rightarrow B^S(H) \rightarrow B^S(H^{\text{tor}}) \rightarrow 0.$$

Set $\bar{B} = \text{im}[B(H^{\text{ssu}}) \rightarrow B(H)]$, then we obtain an exact sequence

$$0 \rightarrow \bar{B} \rightarrow B^S(H) \rightarrow B^S(H^{\text{tor}}) \rightarrow 0$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{B} & \longrightarrow & B^S(H) & \longrightarrow & B^S(H^{\text{tor}}) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{B} & \longrightarrow & B'(H) & \longrightarrow & B'(H^{\text{tor}}) \longrightarrow 0 \end{array}$$

Now the snake lemma gives us an isomorphism $C_S(H) = B'(H)/B^S(H) \xrightarrow{\sim} B'(H^{\text{tor}})/B^S(H^{\text{tor}}) = C_S(H^{\text{tor}})$. \square

Corollary 3.8. *Let k , X , G , H be as in 3.1 (in particular G is quasi-trivial and H is connected). Assume that $H^{\text{tor}} = 1$. Then X has the weak approximation property.*

Proof. By Proposition 3.7 we have $C_S(H) = C_S(H^{\text{tor}}) = 0$ for any S . By Theorem 3.5 X has (WA_S) for any S . \square

Remark. In the case when G is semisimple simply connected, this result was proved in [B1, Corollary 1.7]. For a simple proof see [CTX, Theorem 3.4(b)].

The following result relates $C_S(H)$ to the Galois cohomology of H^{tor} .

Proposition 3.9. *Let T be a k -torus over a number field k . Let $S \subset \mathcal{V}$ be a finite set of places of k . Then there is a canonical isomorphism*

$$C_S(T) \xrightarrow{\sim} \text{coker} \left[H^1(k, T) \rightarrow \prod_{v \in S} H^1(k_v, T) \right].$$

Proof. We have canonical duality isomorphisms

$$\beta_v: H^1(k_v, T) \xrightarrow{\sim} \text{Hom}(H^1(k_v, \mathbf{X}(\bar{T})), \mathbb{Q}/\mathbb{Z}) = B_v(T),$$

cf. [M, Chapter I, Corollary 2.3 and Theorem 2.13]. Moreover, we have an exact sequence

$$(2) \quad H^1(k, T) \xrightarrow{\text{loc}} \oplus_{v \in \mathcal{V}} H^1(k_v, T) \xrightarrow{\mu} B(T),$$

where loc is the localization map, $\mu((\xi_v)_{v \in \mathcal{V}}) = \sum \mu_v(\xi_v)$, and μ_v is the composed map

$$\mu_v: H^1(k_v, T) \xrightarrow{\beta_v} B_v(T) \xrightarrow{\lambda_v} B(T),$$

cf. [M, Chapter I, Theorem 4.20(b)].

Consider the localization map $\text{loc}_S: H^1(k, T) \rightarrow \prod_{v \in S} H^1(k_v, T)$. Let $\xi_S = (\xi_v)_{v \in S} \in \prod_{v \in S} H^1(k_v, T) = \oplus_{v \in S} B_v(T)$, where we identify $H^1(k_v, T)$

with $B_v(T)$ using β_v . From exact sequence (2) we see that $\xi_S \in \text{im loc}_S$ if and only if there exists an element $\xi^S \in \bigoplus_{v \notin S} B_v(T)$ such that $\mu(\xi_S, \xi^S) = 0$. Such an element ξ^S exists if and only if

$$\sum_{v \in S} \mu_v(\xi_v) \in B^S(T) \subset B(T).$$

Set $B_S(T) = \langle \lambda_v(B_v(T)) \rangle_{v \in S}$. Then we see that there is a canonical isomorphism

$$\begin{aligned} \text{coker} \left[H^1(k, T) \rightarrow \prod_{v \in S} H^1(k_v, T) \right] &\xrightarrow{\sim} B_S(T) / (B_S(T) \cap B^S(T)) \simeq \\ &\simeq (B_S(T) + B^S(T)) / B^S(T) = B'(T) / B^S(T) = C_S(T). \end{aligned}$$

□

Proposition 3.10. *Let T be a k -torus over a number field k . Let L/k be a Galois extension splitting T . Let S_0 be the set of (nonarchimedean, ramified in L) places v of k whose decomposition groups in $\text{Gal}(L/k)$ are noncyclic. Let $S \subset \mathcal{V}$ be any finite set of places of k . Then the canonical homomorphism $C_S(T) \rightarrow C_{S \cap S_0}(T)$ is an isomorphism.*

Proof. Let $v \in S$. Let w be a place of L lying over v . Let $D_w \subset \text{Gal}(L/k)$ be the decomposition group of w . Then by [Sa, Lemme 6.9] $\text{Pic}(T_{k_v}) = H^1(D_w, \mathbf{X}(T_L))$. We see that the image $\lambda_v(B_v(T)) \subset B(T)$ depends only on the conjugacy class of $D_w \subset \text{Gal}(L/k)$.

If $v \in S$, $v \notin S_0$, then D_w is cyclic for w lying over v . By Chebotarev's density theorem there exists $v' \notin S$ and w' lying over v' such that $D_{w'} = D_w$. It follows that $\lambda_v(B_v(T)) = \lambda_{v'}(B_{v'}(T))$. But $v' \notin S$, hence $\lambda_{v'}(B_{v'}(T)) \subset B^S(T)$. We see that $\lambda_v(B_v(T)) \subset B^S(T)$. Thus $B^{S \cap S_0}(T) = B^S(T)$. We conclude that $C_{S \cap S_0}(T) = C_S(T)$. □

Corollary 3.11. *Let H be a connected linear k -group over a number field k . Let L/k be a Galois extension splitting H^{tor} . Let S_0 be the set of places v of k whose decomposition groups in $\text{Gal}(L/k)$ are noncyclic. Let $S \subset \mathcal{V}$ be any finite set of places of k . Then the canonical homomorphism $C_S(H) \rightarrow C_{S \cap S_0}(H)$ is an isomorphism.*

Proof. We have a commutative diagram of canonical homomorphisms

$$\begin{array}{ccc} C_S(H) & \xrightarrow{\simeq} & C_S(H^{\text{tor}}) \\ \downarrow & & \downarrow \simeq \\ C_{S \cap S_0}(H) & \xrightarrow{\simeq} & C_{S \cap S_0}(H^{\text{tor}}) \end{array}$$

By Proposition 3.7 the horizontal arrows are isomorphisms. By Proposition 3.10 the right vertical arrow is an isomorphism. We conclude that the left vertical arrow is also an isomorphism. □

Theorem 3.12. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Let L/k be a Galois extension splitting H^{tor} . Let S_0 be the set of places v of k whose decomposition groups in $\text{Gal}(L/k)$ are noncyclic. Let $S \subset \mathcal{V}$ be a finite set of places of k such that $S \cap S_0 = \emptyset$. Then X has (WA_S) .*

Proof. By Corollary 3.11 $C_S(H) = C_{S \cap S_0}(H) = C_\emptyset(H) = 0$. By Theorem 3.5 X has (WA_S) . \square

Corollary 3.13. *Let X be as in 3.1. Then X has the real approximation property.*

Proof. Let L and S_0 be as in Theorem 3.12. Take $S = \mathcal{V}_\infty$. A decomposition group of an archimedean place is either 0 or $\mathbb{Z}/2$, hence cyclic. We see that $\mathcal{V}_\infty \cap S_0 = \emptyset$. By Theorem 3.12 X has $(\text{WA}_{\mathcal{V}_\infty})$, i.e. X has the real approximation property. \square

Another proof. The subgroup H is connected, and we have $\text{III}(G) = 0$ and $A(G) = 0$. Now by [B2, Corollary 1.7] X has the real approximation property. \square

Corollary 3.14. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Assume that H^{tor} splits over a cyclic extension L of k . Then X has the weak approximation property.*

Proof. Let S_0 denote the set of places v of k whose decomposition groups in $\text{Gal}(L/k)$ are noncyclic, then $S_0 = \emptyset$. Thus for any finite $S \subset \mathcal{V}$ we have $S \cap S_0 = \emptyset$. By Theorem 3.12 X has (WA_S) for any S , i.e. X has the weak approximation property. \square

4. METACYCLIC EXTENSIONS

In this section, inspired by [Sa, Lemme 1.3], we generalize Corollary 3.14.

4.1. Recall that a finite group is called metacyclic if all its Sylow subgroups are cyclic. For example, the symmetric group S_3 is metacyclic, while the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is not. Every cyclic group is metacyclic. We say that a Galois extension L/k is metacyclic if $\text{Gal}(L/k)$ is a metacyclic group.

Theorem 4.2. *Let k, X, G, H be as in 3.1 (in particular G is quasi-trivial and H is connected). Assume that H^{tor} splits over a metacyclic extension L of k . Then X has the weak approximation property.*

Proof. Set $T = H^{\text{tor}}$, then T is a k -torus splitting over L . By Theorem 3.5 and Proposition 3.7 it suffices to prove that $C_S(T) = 0$. Set

$$\mathcal{U}_S^1(T) = \text{coker} \left[H^1(k, T) \rightarrow \prod_{v \in S} H^1(k_v, T) \right].$$

By Proposition 3.9 $C_S(T) \simeq \mathcal{U}_S^1(T)$.

We write \hat{T} for $\mathbf{X}(\overline{T})$. Set

$$\begin{aligned}\mathrm{III}_S^1(k, \hat{T}) &= \ker \left[H^1(k, \hat{T}) \rightarrow \prod_{v \in \mathcal{V} \setminus S} H^1(k_v, \hat{T}) \right] \\ \mathrm{III}_\omega^1(k, \hat{T}) &= \bigcup_S \mathrm{III}_S^1(k, \hat{T}) \\ \mathrm{III}_{S, \emptyset}^1(k, \hat{T}) &= \mathrm{III}_S^1(k, \hat{T}) / \mathrm{III}_\emptyset^1(k, \hat{T}).\end{aligned}$$

By [Sch, Theorem 4.2]

$$\mathfrak{U}_S^1(T) \simeq \mathrm{Hom}(\mathrm{III}_{S, \emptyset}^1(k, \hat{T}), \mathbb{Q}/\mathbb{Z}).$$

Now $\mathrm{III}_{S, \emptyset}^1(k, \hat{T})$ is by definition a subquotient of $\mathrm{III}_\omega^1(k, \hat{T})$. Thus in order to prove the theorem it suffices to show that $\mathrm{III}_\omega^1(k, \hat{T}) = 0$.

Denote by \mathfrak{g} the image of $\mathrm{Gal}(L/k)$ in $\mathrm{Aut}(\hat{T})$. Then \mathfrak{g} is a finite metacyclic group. We may and shall assume that $\mathrm{Gal}(L/k) = \mathfrak{g}$. For a place v of k , let $D_w \subset \mathfrak{g}$ denote the decomposition group of a place w of L extending v . We write \mathfrak{g}_v for D_w , it is defined up to conjugacy in \mathfrak{g} . Since $\mathrm{Gal}(\overline{k}/L)$ is a profinite group and \hat{T} is a free abelian group, we have $H^1(L, \hat{T}) = 0$. It follows that the inflation homomorphism $H^1(\mathfrak{g}, \hat{T}) \rightarrow H^1(k, \hat{T})$ is an isomorphism. Similarly, for each v the homomorphism $H^1(\mathfrak{g}_v, \hat{T}) \rightarrow H^1(k_v, \hat{T})$ is an isomorphism. Thus we obtain an isomorphism

$$\ker \left[H^1(\mathfrak{g}, \hat{T}) \rightarrow \prod_{v \in \mathcal{V} \setminus S} H^1(\mathfrak{g}_v, \hat{T}) \right] \xrightarrow{\sim} \mathrm{III}_S^1(k, \hat{T}).$$

It follows from Chebotarev's density theorem that

$$\mathrm{III}_\omega^1(k, \hat{T}) \simeq \ker \left[H^1(\mathfrak{g}, \hat{T}) \rightarrow \prod_C H^1(C, \hat{T}) \right],$$

where C runs over all cyclic subgroups of \mathfrak{g} .

Now let \mathfrak{g} be any finite group and Y a finitely generated \mathfrak{g} -module. Let $i \in \mathbb{Z}$. We write $\hat{H}^i(\mathfrak{g}, Y)$ for the i -th Tate cohomology group. Following an idea of [CTK, page 734], we set

$$\mathrm{III}_\Omega^i(\mathfrak{g}, Y) = \ker \left[\hat{H}^i(\mathfrak{g}, Y) \rightarrow \prod_C \hat{H}^i(C, Y) \right],$$

where C runs over all cyclic subgroups of \mathfrak{g} . Then $\mathrm{III}_\omega^1(k, \hat{T}) \simeq \mathrm{III}_\Omega^1(\mathfrak{g}, \hat{T})$. In order to prove Theorem 4.2 it suffices to show that $\mathrm{III}_\Omega^1(\mathfrak{g}, \hat{T}) = 0$, which follows from the next lemma. \square

Lemma 4.3 (B. Konyavskiĭ, private communication). *Let \mathfrak{g} be a finite group and Y a finitely generated \mathfrak{g} -module. If \mathfrak{g} is metacyclic, then $\mathrm{III}_\Omega^i(\mathfrak{g}, Y) = 0$ for all $i \in \mathbb{Z}$.*

Proof. Let $y \in \text{III}_\Omega^i(\mathfrak{g}, Y) \subset \hat{H}^i(\mathfrak{g}, Y)$. For a subgroup $\mathfrak{h} \subset \mathfrak{g}$ let $\text{Res}_\mathfrak{h}(y) \in \hat{H}^i(\mathfrak{h}, Y)$ denote the restriction of y to \mathfrak{h} . Since $y \in \text{III}_\Omega^i(\mathfrak{g}, Y)$, we have $\text{Res}_C(y) = 0$ for any cyclic subgroup $C \subset \mathfrak{g}$. Since \mathfrak{g} is metacyclic, every Sylow subgroup of \mathfrak{g} is cyclic. We see that $\text{Res}_S(y) = 0$ for any Sylow subgroup S of \mathfrak{g} . By [ANT, Chapter IV, Section 6, Corollary 4 of Proposition 8] we have $y = 0$. Thus $\text{III}_\Omega^i(\mathfrak{g}, Y) = 0$. This completes the proofs of the lemma and of Theorem 4.2. \square

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